Parliament Seating Ideas

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October 21, 2021

1 Base model

The model proposed by Vangerven et al. [3] is:

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{p \in P, e \in E} w_e y_{ep} && && \text{(1a)}\\
\textbf{subject to} \quad & \sum_{p \in P} x_{ip} \leq 1 && \forall i \in V, && \text{(1b)}\\
& \sum_{i \in V} x_{ip} = \beta^p && \forall p \in P, && \text{(1c)}\\
& y_{ep} \leq x_{ip} && \forall p \in P, e \in E, i \in e, && \text{(1d)}\\
& \sum_{i \in F} x_{ip} \geq r^p && \forall p \in P, && \text{(1e)}\\
& x \text{ induces connected components}, && && \text{(1f)}\\
& x_{ip} \in \{0, 1\} && \forall i \in V, p \in P, && \text{(1g)}\\
& y_{ep} \in \{0, 1\} && \forall e \in E, p \in P. && \text{(1h)}
\end{aligned}
$$

Here, y assigns edges to parties, x assigns vertices to parties, β^p is the target size (in number of vertices) of party p, r^p is the target number of front-row seats of party p, and $R \subseteq V$ is the set of front-row seats.

2 Lagrangean-based fixing

If we relax constraints (1b) and (1d), the objective function of the corresponding Lagrangean with multiplier vectors α and λ, respectively, is:

$$
\begin{aligned}
& \sum_{p \in P, e \in E} w_e y_{ep} + \sum_{i \in V} \alpha_i (1 - \sum_{p \in P} x_{ip}) + \sum_{p \in P, e \in E, i \in e} \lambda^i_{ep} (x_{ip} - y_{ep}) \\
= & \sum_{i \in V} \alpha_i + \sum_{p \in P, e \in E} y_{ep} (w_e - \sum_{i \in e} \lambda^i_{ep}) - \sum_{i \in V, p \in P} \alpha_i x_{ip} + \sum_{p \in P, e \in E, i \in E} \lambda^i_{ep} x_{ip}.
\end{aligned}
$$

Expression $\sum_{p \in P, e \in E, i \in E} \lambda^i_{ep} x_{ip}$ can be written as $\sum_{i \in V, p \in P} x_{ip} \sum_{e \in N(i)} \lambda^i_{ep}$, and the above becomes

$$
\sum_{i \in V} \alpha_i + \sum_{p \in P, e \in E} y_{ep} (w_e - \sum_{i \in e} \lambda^i_{ep}) + \sum_{p \in P, i \in V} x_{ip} (\sum_{e \in N(i)} \lambda^i_{ep} - \alpha_i).
$$

Let $\hat{w}_{ep} = w_e - \sum_{i \in e} \lambda^i_{ep}$ for $e \in E, p \in P$, $\hat{w}_{ip} = \sum_{e \in N(i)} \lambda^i_{ep} - \alpha_i$ for $i \in V, p \in P$, and $\hat{\alpha} = \sum_{i \in V} \alpha_i$ be constants wrt. Lagrangean multiplier vectors α, λ. Then, ignoring connectivity constraints (1f), the Lagrangean relaxation becomes:

$$
\begin{aligned}
\textbf{maximize} \quad & \hat{\alpha} + \sum_{p \in P, e \in E} \hat{w}_{ep} y_{ep} + \sum_{p \in P, i \in V} \hat{w}_{ip} x_{ip} & & \text{(2a)} \\
\textbf{subject to} \quad & \sum_{i \in V} x_{ip} = \beta^p & \forall p \in P, \quad & \text{(2b)} \\
& \sum_{i \in F} x_{ip} \geq r^p & \forall p \in P, \quad & \text{(2c)} \\
& x_{ip} \in \{0, 1\} & \forall i \in V, p \in P, \quad & \text{(2d)} \\
& y_{ep} \in \{0, 1\} & \forall e \in E, p \in P. \quad & \text{(2e)}
\end{aligned}
$$

This can be decomposed into p independent subproblems. Each subproblem can be solved in $O(m + n \log n)$ time by setting $y_{ep} = [\hat{w}_{ep} \geq 0]$ and then selecting, for each party p, first r^p vertices from F with maximum $\hat{w}_{\cdot p}$, followed by $\beta^p - r^p$ not-yet-selected vertices from V with maximum $\hat{w}_{\cdot p}$, and setting $x_{\cdot p} = 1$ for these vertices.

Let an optimal solution of cost U to (2) be denoted as (x^*, y^*), and let $S_p = \{i \in V \mid x^*_{ip} = 1\}$. We can then compute upper bounds U^x_{ip} and U^y_{ep} obtained by tentatively flipping x^*_{ip} or y^*_{ep} as follows:

$$
U^x_{ip} = \begin{cases}
U + \hat{w}_{ip} - \begin{cases} \infty & \text{if } |S_p| = r^p \\ \min_{j \in S_p} \hat{w}_{jp} & \text{if } |S_p \cap F| + [i \in F] > r^p \\ \min_{j \in S_p \setminus F} \hat{w}_{jp} & \text{otherwise} \end{cases} & \text{if } x^*_{ip} = 0 \\
U - \hat{w}_{ip} + \begin{cases} -\infty & \text{if } |F| = r^p \\ \max_{j \in V \setminus S_p} \hat{w}_{jp} & \text{if } |S_p \cap F| - [i \in F] \geq r^p \\ \max_{j \in F \setminus S_p} \hat{w}_{jp} & \text{otherwise} \end{cases} & \text{if } x^*_{ip} = 1
\end{cases}
$$

$$
U^y_{ep} = \begin{cases} U + \hat{w}_{ep} & \text{if } y^*_{ep} = 0 \\ U - \hat{w}_{ep} & \text{if } y^*_{ep} = 1. \end{cases}
$$

Given some lower bound L to model (1) (obtained e.g. by a heuristic), for any $i \in V, p \in P$ such that $U^x_{ip} < L$ we can fix $x_{ip} = 1 - x^*_{ip}$ in the original MIP. Similarly, for any $e \in E, p \in P$ such that $U^y_{ep} < L$ we fix $y_{ep} = 1 - y^*_{ep}$. Note that each U^x_{ip} can be computed in amortized constant time by pre-computing, for each p, the results of sub-cases 2 and 3.

3 Alternative formulation minimizing cut edges

Alternatively, we can write model (1) in order to minimize the cost of cut edges, since the solution on the x variables is equivalent. Borrowing linking constraints (3b) from Validi et al.

[2], we have:

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{p\in P, e\in E} w_e \tilde{y}_{ep} && && (3a)\\
\textbf{subject to} \quad & (1b), (1c), (1f), (1e), (1g), && \\
& x_{ip} - x_{jp} \leq \tilde{y}_{ep} && \forall p \in P, e = \{i,j\} \in E, && (3b)\\
& \tilde{y}_{ep} \in \{0,1\} && \forall e \in E, p \in P, && (3c)
\end{aligned}$$

where $\tilde{y}_{ep}$ is equivalent to $1 - y_{ep}$ and denotes whether an edge it not part of party p.

In practice, this model seems to fare a little better than the original model. Validi et al. [2] also suggest that adding the complementary constraints $x_{jp} - x_{ip} \leq \tilde{y}_e$ to constraints (3b) could improve LP bounds, but in practice it seems this doesn't help much.

We can also write a Lagrangean objective for this formulation, relaxing (1b) and 3b:

$$\begin{aligned}
& \sum_{p\in P, e\in E} w_e \tilde{y}_{ep} + \sum_{i\in V} \alpha_i (1 - \sum_{p\in P} x_{ip}) + \sum_{p\in P, e=\{i,j\}\in E} \mu_{ep}(\tilde{y}_{ep} - x_{ip} + x_{jp}) \\
= & \sum_{p\in P, e\in E} \tilde{y}_{ep}(w_e + \mu_{ep}) + \sum_{i\in V} \alpha_i - \sum_{i\in V}\sum_{p\in P} x_{ip}(\alpha_i + \sum_{e=\{i,j\}\in N(i)} \mu_{ep}(2 * [i > j] - 1));
\end{aligned}$$

Letting $\tilde{w}_{ep} = w_e + \mu_{ep}$ for $e \in E, p \in P$, $\tilde{w}_{ip} = \alpha_i + \sum_{e=\{i,j\}\in N(i)} \mu_{ep}(2 * [i > j] - 1)$ for $i \in V, p \in P$ and $\tilde{\alpha} = \sum_{i\in V} \alpha_i$, we can write the Lagrangean as:

$$\begin{aligned}
\textbf{minimize} \quad & \tilde{\alpha} + \sum_{p\in P, e\in E} \tilde{w}_{ep} y_{ep} + \sum_{p\in P, i\in V} \tilde{w}_{ip} x_{ip} && (4a)\\
\textbf{subject to} \quad & (2b), (2c), (2d), (2e) && (4b)
\end{aligned}$$

The solution is analogous to that of model (2), but we select smallest instead of largest.

It would be interesting to see whether linking constraints (3b) are stronger than (1d), and how we can translate them directly to the maximization model.

4 Connectivity

Vangerven et al. [3] use multi-commodity flow (MCF) constraints to model connectivity, but Hojny et al. [1] propose a single-commodity flow (SCF) formulation. In practice, SCF seems better.

References

[1] Christopher Hojny et al. "Mixed-integer programming techniques for the connected max-k-cut problem". In: *Mathematical Programming Computation* 13.1 (2021), pp. 75–132.

[2] Hamidreza Validi et al. "Political districting to minimize cut edges". In: *Optimization Online* (2021). URL: http://www.optimization-online.org/DB_HTML/2021/04/8349.html.

[3] Bart Vangerven et al. "Parliament seating assignment problems". In: *European Journal of Operational Research* (2021).